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Concavity of the CES function via the power mean inequality

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Abstract

In this note, we analyze the concavity, and the convexity, of the constant elasticity of substitution (CES) function by means of the power mean inequality.

Keywords: CES function, concavity, gradient inequality, power mean inequality.
JEL Classification: D10, D20.

1. Introduction

To prove the concavity, or convexity, of the constant elasticity of substitution (CES) utility, or production, function (see Arrow et al., 1961), one can check when the associated hessian matrix is negative, or positive, semidefinite (see, for instance, Simon and Blume, 1994, Theorem 21.5, p. 513). This general procedure, however, may be rather cumbersome when the number of commodities is large. Therefore, an alternative, more straightforward, proof would be useful, also in the light of the fact that the CES function is widely used in economic models.

In this note, we use the power mean inequality to show when the CES function satisfies the gradient inequality that characterizes a concave, or a convex, function. Our approach has both, advantages and disadvantages: on the one hand, assuming a smaller or larger number of commodities does not affect the application of our proof, which is a rather immediate consequence of the power mean inequality. On the other hand, the proof is tailored to the CES function, and cannot be directly extended to other functional forms.

The relation between the CES function and the power mean is well known since Arrow et al. (1961); exploiting it to study the concavity, and the convexity, of the function is, to the best of our knowledge, an original contribution. Actually, our proof was inspired by Afriat (1987, p. 189), who shows that the Cobb-Douglas utility, or production, function is concave by verifying

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that it satisfies the gradient inequality that characterizes a concave function. The proof is based on a direct application of the arithmetic mean-geometric mean inequality, which is a special case of the power mean inequality. Therefore, our result can be regarded as a generalization of Afriat's approach to the case of the CES function.

2. The main result

Let $\gamma \in \mathbb{R}$, with $\gamma \neq 0$, $X = \mathbb{R}^J_{++}$, and $a_j > 0$, $j = 1, 2, ...J$.[1] Consider the constant elasticity of substitution function $f : X \to \mathbb{R}$ defined as

$$f(x) = \left(\sum_{j=1}^{J} a_j x_j^\gamma \right)^{\frac{1}{\gamma}}$$

When $\gamma = 1$, f is linear, hence both concave and convex on X. In what follows, we concentrate on the case $\gamma \neq 1$.

Recall that a differentiable real-valued function $\tilde{f}$ defined on an open convex set X in $\mathbb{R}^J$ is concave if and only if, for every $x, \hat{x} \in X$,

$$\tilde{f}(x) \leqslant \tilde{f}(\hat{x}) + \nabla \tilde{f}(\hat{x}) \cdot (x - \hat{x}) , \tag{Conc}$$

and it is convex if and only if the above inequality is reversed (see, for instance, Simon and Blume, 1994, Theorem 21.2, p. 510).

Since f is homogeneous of degree one, $f(x) = \nabla f(x) \cdot x$ by the Euler's theorem (see, for instance, Simon and Blume, 1994, Theorem 20.4, p. 491). Therefore, condition (Conc) reduces to

$$f(x) \leqslant \nabla f(\hat{x}) \cdot x \quad \text{for all } x, \hat{x} \in X \tag{1}$$

Let $\varphi(\hat{x}) = \sum_j a_j \hat{x}_j^\gamma$. By direct computation,

$$\frac{\partial f(\hat{x})}{\partial x_j} = \left(\frac{a_j \hat{x}_j^\gamma}{\varphi(\hat{x})} \right) \frac{f(\hat{x})}{\hat{x}_j} ,$$

for every $j = 1, ..., J$, so that

$$\nabla f(\hat{x}) \cdot x = \left[\sum_{j=1}^{J} \left(\frac{a_j \hat{x}_j^\gamma}{\varphi(\hat{x})} \right) \frac{x_j}{\hat{x}_j} \right] f(\hat{x}) \tag{2}$$

[1] It is not required that $a_1 + a_2 + ...a_J = 1$.

Since $f(\hat{x}) > 0$, we can use (2) to rewrite condition (1) as follows:

$$\frac{f(x)}{f(\hat{x})} \leqslant \sum_{j=1}^{J} \left(\frac{a_j \hat{x}_j^{\gamma}}{\varphi(\hat{x})} \right) \frac{x_j}{\hat{x}_j} \tag{3}$$

To show that f is concave, we have to verify that it satisfies (3) for any $x, \hat{x} \in X$. Similarly, to show that it is convex we need to verify that, for any $x, \hat{x} \in X$,

$$\frac{f(x)}{f(\hat{x})} \geqslant \sum_{j=1}^{J} \left(\frac{a_j \hat{x}_j^{\gamma}}{\varphi(\hat{x})} \right) \frac{x_j}{\hat{x}_j} \tag{4}$$

Our proof relies on the power mean inequality, which states that

$$\left(\sum_{j=1}^{J} w_j z_j^r \right)^{\frac{1}{r}} \leqslant \left(\sum_{j=1}^{J} w_j z_j^s \right)^{\frac{1}{s}}$$

for any $r, s \in \mathbb{R}$, $r < s$, and $z_j, w_j > 0,\ j = 1, 2, ...J$, with $w_1 + w_2 + ...w_J = 1$ (see, for instance, Steele, 2004, chapter 8.3). Notice that this inequality implies

$$\left(\sum_{j=1}^{J} w_j z_j^r \right)^{\frac{1}{r}} \leqslant \sum_{j=1}^{J} w_j z_j \tag{5}$$

when $r < 1 = s$, and

$$\sum_{j=1}^{J} w_j z_j \leqslant \left(\sum_{j=1}^{J} w_j z_j^s \right)^{\frac{1}{s}} \tag{6}$$

when $r = 1 < s$. To take the final step, consider the auxiliary variables

$$w_j = \frac{a_j \hat{x}_j^{\gamma}}{\varphi(\hat{x})} \quad \text{and} \quad z_j = \frac{x_j}{\hat{x}_j},$$

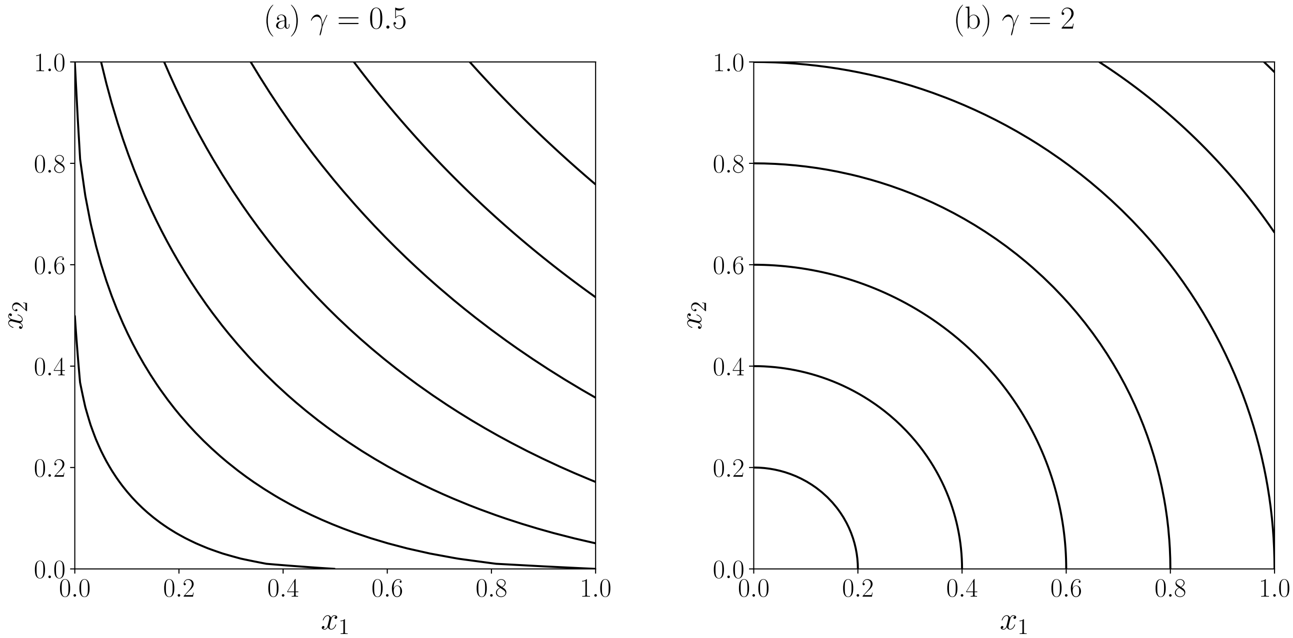
so that $z_j, w_j > 0$ for every j, and $w_1 + w_2 + ...w_J = 1$, and observe that

$$\frac{f(x)}{f(\hat{x})} = \frac{\left(\sum_{j=1}^{J} a_j x_j \right)^{\frac{1}{\gamma}}}{\left(\sum_{j=1}^{J} a_j \hat{x}_j \right)^{\frac{1}{\gamma}}} = \frac{\left(\sum_{j=1}^{J} a_j x_j \right)^{\frac{1}{\gamma}}}{(\varphi(\hat{x}))^{\frac{1}{\gamma}}} = \left(\sum_{j=1}^{J} \frac{a_j \hat{x}_j^{\gamma}}{\varphi(\hat{x})} \frac{x_j}{\hat{x}_j} \right)^{\frac{1}{\gamma}}$$

and also that

$$\sum_{j=1}^{J} \left(\frac{a_j \hat{x}_j^{\gamma}}{\varphi(\hat{x})} \right) \frac{x_j}{\hat{x}_j} = \sum_{j=1}^{J} w_j z_j$$

Figure 1: Curve levels for different values of γ



Therefore, when $\gamma < 1$, (3) is equivalent to (5), hence f is concave on X; when $\gamma > 1$, (5) is equivalent to (6), hence f is convex on X. The proof is thus complete.

Figure 1 shows the curve levels of the CES function when $J = 2$, i.e., when there are two commodities x_1 and x_2. In the left panel, we let $\gamma = 0.5$, so that the function is concave; in the right panel, we let $\gamma = 2$, hence the function is convex.[2]

For sake of completeness, we reproduce the original result due to Afriat (1987), where the function $f : X \to \mathbb{R}$ is defined by[3]

$$f(x) = \prod_{j=1}^{J} x_j^{a_j} \ ,$$

so that

$$\frac{\partial f(\hat{x})}{\partial x_j} = \left(\frac{a_j}{\hat{x}_j}\right) f(\hat{x})\,, \quad \text{hence} \quad \nabla f(\hat{x}) \cdot x = \left[\sum_{j=1}^{J} a_j \left(\frac{x_j}{\hat{x}_j}\right)\right] f(\hat{x})$$

Condition (1) is, therefore, equivalent to

$$\frac{f(x)}{f(\hat{x})} \leqslant \sum_{j=1}^{J} a_j \left(\frac{x_j}{\hat{x}_j}\right) \tag{7}$$

[2]In both figures, it is assumed that $a_1 = a_2 = 1$.

[3]This corresponds to the case $\gamma = 0$, or more precisely $\lim \gamma \to 0$, in the CES function. For this case, it is assumed that $a_1 + a_2 + ...a_J = 1$.

Setting $w_j = a_j$, and z_j as before, implies that

$$\frac{f(x)}{f(\hat{x})} = \left(\prod_{j=1}^{J} \hat{x}_j^{a_j}\right)^{-1} \left(\prod_{j=1}^{J} x_j^{a_j}\right) = \prod_{j=1}^{J} z_j^{w_j} \leqslant \sum_{j=1}^{J} w_j z_j = \sum_{j=1}^{J} a_j \left(\frac{x_j}{\hat{x}_j}\right) ,$$

where the inequality follows from the arithmetic mean-geometric mean inequality, a special case of the power mean inequality; hence f is concave.

3. Conclusion

In this note, we have proposed a straightforward proof to verify the concavity, and the convexity, of the CES function, which is inspired by a proof due to Afriat (1987). It should be contrasted with the general procedure based on the semidefiniteness of the hessian matrix of the function, which becomes cumbersome when the number of commodities is large. However, the proof is tailored to the CES function, and cannot be directly extended to other functional forms.

References

[1] Afriat, S. N. (1987): *Logic of Choice and Economic Theory*. Clarendon Press.

[2] Arrow, K. J., H. B. Chenery, B. S. Minhas, and R. M. Solow (1961): "Capital-labor substitution and economic efficiency", *The Review of Economics and Statistics*, Vol. 43, No. 3, pp. 225–250.

[3] Simon, C. P. and L. Blume (1994): *Mathematics for Economists*. Norton New York.

[4] Steele, J. M. (2004): *The Cauchy-Schwarz Master Class*. Cambridge University Press.